Webology (ISSN: 1735-188X)
Volume 18, Number 6, 2021

Fekete-Szegö Inequality For Subclasses Of A New Class Of Analytic Functions Involving Integral Operator

Preet Pal Singh[1] , Gurmeet Singh[2]

[1]Pt. L. M.S. Govt. P.G. College Rishikesh, Dehradun, U.K.)

[2]GSSDGS Khalsa college , Patiala)

Abstract: We introduce a new class of analytic functions and its subclasses and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes.

Keywords: bounded functions, Inverse Starlike functions, Starlike functions and Univalent functions.

Mathematics subject classification: 30C45, 30C50

I. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([8], [9]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [7] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö [10] used Löwner's method to prove the followingwell known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, \text{if } \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 3, \text{if} \mu \geq 1. \end{cases} \tag{1.2}$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (Chhichra[1], Babalola[7]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ satisfying the condition

$$\mathrm{Re}\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \quad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{A}$ satisfying the condition

$$\mathrm{Re}\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \quad (1.4)$$

Gurmeet Singh, Saroa M. S. and Mehrok, B. S. [4] have introduced the class of Inverse Starlike functions as the functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$$\mathrm{Re}\left(\frac{zf(z)}{2\int_0^z f(z)dz}\right) > 0, z \in \mathbb{E} \ \text{ i.e.} \frac{zf(z)}{2\int_0^z f(z)dz} \prec \frac{1+z}{1-z} \quad (1.5)$$

[4] denoted this class by $(S^*)^{-1}$.

The subclass of $(S^*)^{-1}$ consisting of the functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$and satisfying the condition

$$\frac{zf(z)}{2\int_0^z f(z)dz} \prec \frac{1+Az}{1+Bz}; -1 \le B \le A \le 1 \quad (1.6)$$

is denoted by $(S^*)^{-1}[A,B]$ (See [4]).

Symbol $\prec$ stands for subordination, which we define as follows:

p-valent functions: A function $f(z) \in \mathcal{A}_p$ is said to be a p-valent function in E if it assumes no value more than p times in E.

The class of functions $f(z) \in \mathcal{A}_p$ satisfying the condition

$$\frac{zf(z)}{(p+1)\int_0^z f(z)dz} \prec \frac{1+z}{1-z}$$

is denoted by $\left(S_p{}^*\right)^{-1}$.(See [4])

These functions were called p- valently inverse starlike functions. In this paper, We will deal with $\left(S_p{}^*\right)^{-1}[A,B]$, the subclass of $\left(S_p{}^*\right)^{-1}$ consisting of the functionsf(z) $\in \mathcal{A}_p$and satisfying the condition

$$\frac{zf(z)}{(p+1)\int_0^z f(z)dz} \prec \frac{1+Az}{1+Bz}; -1 \le B \le A \le 1.$$

We will also deal with $\left(S_p{}^*\right)^{-1}[A,B;\delta]$, the subclass of $\left(S_p{}^*\right)^{-1}[A,B]$ consisting of the functionsf(z) $\in \mathcal{A}$and satisfying the condition

$$\frac{zf(z)}{(p+1)\int_0^z f(z)dz} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}; -1 \le B \le A \le 1;\ \delta > 0.$$

We will establish Fekete-Szegö inequality for these classes.

Principle of subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to $F(z)$ in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1$. (1.7)

It is known that $|d_1| \le 1, |d_2| \le 1 - |d_1|^2$.

1. Main Results

Theorem 2.1: If $f(z)\in\left(S_p{}^*\right)^{-1}[A,B]$, then

$$\frac{1}{A-B}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \begin{cases} (p+2)^2(A-B)\left(\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}-\mu\right), \\ \text{if } \mu\leq\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)} \ (2.1) \\ \frac{p+3}{2}, \\ \text{if } \frac{(p+3)[\{(p+1)(A-B)-B\}-1]}{2(p+2)^2(A-B)}\leq\mu\leq\frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)} \ (2.2) \\ (p+2)^2(A-B)\left(\mu-\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}\right), \\ \text{if } \mu\geq\frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)} \ (2.3) \end{cases}$$

The results are sharp.

Proof: By definition of $\left(S_p{}^*\right)^{-1}[A,B]$, we have

$$\frac{zf(z)}{(p+1)\int_0^z f(z)dz}=\left(\frac{1+Aw(z)}{1+Bw(z)}\right);-1\leq B\leq A\leq 1, \quad (2.4)$$

Expanding (2.4), we have

$$\left(1+a_{p+1}z+a_{p+2}z^2+---\right) = \left(1+\frac{p+1}{p+2}a_{p+1}z+\frac{p+1}{p+3}a_{p+2}z^2+---\right)\left(1+(A-B)c_1z+(A-B)(c_2-Bc_1^2)z^2+---\right) \quad (2.5)$$

Identifying terms in (2.5), we get

$a_{p+1}=(p+2)(A-B)c_1$ and

$a_{p+2}=\frac{(p+3)(A-B)}{2}[c_2+\{(p+1)(A-B)-B\}c_1^2]$ (2.6)

Using (2.5) and (2.6), we obtain

$$a_{p+2}-\mu a_{p+1}^2=\frac{(p+3)(A-B)}{2}c_2+(p+2)^2(A-B)^2\left[\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}-\mu\right]c_1^2$$

$$\frac{1}{A-B}\left(a_{p+2}-\mu a_{p+1}^2\right)=\frac{(p+3)}{2}c_2+(p+2)^2(A-B)\left[\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}-\mu\right]c_1^2$$

This leads to

$$\frac{1}{A-B}\left|a_{p+2}-\mu a_{p+1}^2\right|\leq\frac{p+3}{2}|c_2|+\ (+2)^2(-B)\left|\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}-\mu\right||c_1|^2$$

$$\leq\frac{p+3}{2}+(p+2)^2(A-B)\left[\left|\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}-\mu\right|-\frac{p+3}{2(p+2)^2(A-B)}\right]|c_1|^2 \quad (2.7)$$

Case I: $\mu \leq \frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}$, we get from (2.7)

$$\frac{1}{A-B}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{p+3}{2}+(p+2)^2(A-B)\left[\frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}-1]}{(p+2)^2(A-B)}-\mu\right]|c_1|^2 \quad (2.8)$$

Subcase I(a): $\mu \leq \frac{(p+3)[\{(p+1)(A-B)-B\}-1]}{2(p+2)^2(A-B)}$. From equation (2.8), we get

$$\frac{1}{A-B}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq (p+2)^2(A-B)\left(\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}-\mu\right). \quad (2.9)$$

Subcase I(b): $\mu \geq \frac{(p+3)[\{(p+1)(A-B)-B\}-1]}{2(p+2)^2(A-B)}$.From equation (2.8), we get

$$\frac{1}{A-B}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{p+3}{2}. \quad (2.10)$$

Case II: $\mu \geq \frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}$, we get from (2.7)

$$\frac{1}{A-B}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{p+3}{2}+(p+2)^2(A-B)\left[\mu-\frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)}\right]|c_1|^2 \quad (2.11)$$

Subcase II(a): $\mu \leq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)}$. From equation (2.11), we get

$$\frac{1}{A-B}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{p+3}{2}. \quad (2.12)$$

Combining subcase I(a) and subcase II(b), we get

$$\frac{1}{A-B}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{p+3}{2}, \text{if } \frac{(p+3)[\{(p+1)(A-B)-B\}-1]}{2(p+2)^2(A-B)} \leq \mu \leq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)} \quad (2.13)$$

Subcase II(b): $\mu \geq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)}$. From equation (2.11), we get

$$\frac{1}{A-B}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq (p+2)^2(A-B)\left(\mu-\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}\right). \quad (2.14)$$

This completes the theorem. The results are sharp.

Extremal function for first and third inequality is

$$f_1(z)=(p+1)z^p(1+Az)(1+Bz)^{\frac{(p+1)(A-B)-B}{B}}$$

Extremal function for second inequality is

$$f_2(z)=(p+1)z^p(1+Az^2)(1+Bz^2)^{\frac{(p+1)(A-B)-2B}{2B}}$$

Corollary 2.2: Putting $A=1, B=-1$ in Theorem 2.1, we get

$$\frac{1}{2}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \begin{cases} 2(p+2)^2\left(\frac{(p+3)(2p+3)}{4(p+2)^2}-\mu\right), \text{if } \mu \leq \frac{(p+3)(2p+3)}{4(p+2)^2}; \\ \frac{p+3}{2}, \text{if } \frac{(p+3)(p+1)}{2(p+2)^2} \leq \mu \leq \frac{p+3}{2(p+2)}; \\ 2(p+2)^2\left(\mu-\frac{(p+3)(2p+3)}{4(p+2)^2}\right), \text{if } \mu \geq \frac{p+3}{2(p+2)} \end{cases}$$

, which are the required results for the class $\left(S_p{}^*\right)^{-1}$(See [4].

Corollary 2.3: Putting $p=1$ in Theorem 2.1, we get

$$\frac{1}{A-B}|a_3-\mu a_2^2| \leq \begin{cases} 9(A-B)\left(\frac{2(2A-3B)}{9(A-B)}-\mu\right), \text{if } \mu \leq \frac{2(2A-3B)}{9(A-B)}; \\ 2, \text{if } \frac{2(2A-3B)}{9(A-B)} \leq \mu \leq \frac{2(2A-3B+1)}{9(A-B)}; \\ 9(A-B)\left(\mu-\frac{2(2A-3B)}{9(A-B)}\right), \text{if } \mu \geq \frac{2(2A-3B+1)}{9(A-B)} \end{cases}$$

, which are the required results for the class $(S^*)^{-1}[A,B]$(See [4].

Corollary 2.4: Putting $A = 1, B = -1, p = 1$ in Theorem 2.1, we get

$$\frac{1}{4}|a_3-\mu a_2^2| \leq \begin{cases} (5-9\mu), \text{if } \mu \leq \frac{4}{9}; \\ 1, \text{if} \frac{4}{9} \leq \mu \leq \frac{2}{3}; \\ (9\mu-5), \text{if } \mu \geq \frac{2}{3}. \end{cases}$$

, which are the required results for the class $(S^*)^{-1}$.(See [4]).

Theorem 2.5: If $\left(S_p{}^*\right)^{-1}[\delta]$, then

$$\frac{1}{2\delta}\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \begin{cases} \delta\left\{\frac{p+3}{2}(2p+3)-2\mu(p+2)^2\right\}, \\ \text{if } \mu \leq \frac{(p+1)(p+3)}{2(p+2)^2}, & (2.15) \\ \frac{p+3}{2}, \\ \text{if} \frac{(p+1)(p+3)}{2(p+2)^2} \leq \mu \leq \frac{(p+3)(2p\delta+3\delta-3)}{4\delta(p+2)^2}; & (2.16) \\ \delta\left\{2\mu(p+2)^2-\frac{p+3}{2}(2p+3)\right\}, \\ \text{if } \mu \geq \frac{(p+3)(2p\delta+3\delta-3)}{4\delta(p+2)^2}. & (2.17) \end{cases}$$

The results are sharp.

Proof: We have

$$\frac{zf(z)}{(p+1)\int_0^z f(z)dz} = \left(\frac{1+w(z)}{1-w(z)}\right)^{\delta}; \delta > 0 \qquad (2.18)$$

Expanding we have

$$\left(1+a_{p+1}z+a_{p+2}z^2+---\right)$$
$$= (1+\frac{p+1}{p+2}a_{p+1}z+\frac{p+1}{p+3}a_{p+2}z^2+---)(1+2\delta\, c_1 z+2\delta[c_2+\delta\, {c_1}^2]z^2--)$$

Identifying terms, we get $a_{p+1} = 2\delta(p+2)c_1$ and

$a_{p+2} = (p+3)\,\delta c_2 + (p+3)(2p+3)\delta^2 c_1^2$(2.19)

Using (2.19), we obtain

$$a_{p+2}-\mu a_{p+1}^2 = (p+3)\delta c_2 + (p+3)(2p+3)\delta^2 c_1^2 - \mu(2\delta(p+2)c_1)^2$$

This gives

$$a_{p+2}-\mu a_{p+1}^2 = (p+3)\delta c_2+\delta^2\{(p+3)(2p+3) - 4\mu(p+2)^2\}c_1^2$$

This gives

$$\frac{1}{2\delta}\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p+3}{2}|c_2|+\delta\left|\frac{p+3}{2}(2p+3)-2\mu(p+2)^2\right||c_1^2|$$
$$\le \frac{p+3}{2}+\left[\delta\left|\frac{p+3}{2}(2p+3)-2\mu(p+2)^2\right|-\frac{p+3}{2}\right]|c_1^2| \quad (2.20)$$

Case I: $\mu \le \frac{\frac{p+3}{2}(2p+3)}{2(p+2)^2}$, we get from (2.20) that

$$\frac{1}{2\delta}\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p+3}{2}+\delta[(p+1)(p+3) - 2\mu(p+2)^2]|c_1^2| \quad (2.21)$$

Subcase I(a): $\mu \le \frac{(p+1)(p+3)}{2(p+2)^2}$, Using $|c_1^2| \le 1$, (2.21) takes the form

$$\frac{1}{2\delta}\left|a_{p+2}-\mu a_{p+1}^2\right| \le \delta\left\{\frac{p+3}{2}(2p+3)-2\mu(p+2)^2\right\} (2.22)$$

Subcase I(b): $\mu \ge \frac{(p+1)(p+3)}{2(p+2)^2}$. (2.21) takes the form

$$\frac{1}{2\delta}\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p+3}{2} \quad (2.23)$$

Case II: $\mu \ge \frac{\frac{p+3}{2}(2p+3)}{2(p+2)^2}$, we get from (2.20) that

$$\frac{1}{2\delta}\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p+3}{2}+\left[2\delta(p+2)^2\mu-\frac{p+3}{2}(2p\delta+3\delta-3)\right]|c_1^2| \quad (2.24)$$

Subcase II (a): $\mu \le \frac{(p+3)(2p\delta+3\delta-3)}{4\delta(p+2)^2}$. (2.24) takes the form

$$\frac{1}{2\delta}\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p+3}{2}. \quad (2.25)$$

Subcase I(b): $\mu \ge \frac{(p+3)(2p\delta+3\delta-3)}{4\delta(p+2)^2}$. Using $|c_1^2| \le 1$, (2.24) takes the form

$$\frac{1}{2\delta}\left|a_{p+2}-\mu a_{p+1}^2\right| \le \delta\left\{2\mu(p+2)^2-\frac{p+3}{2}(2p+3)\right\} \quad (2.26)$$

This completes the theorem. The results are sharp.

Corollary 2.6: Putting $\delta = 1$ in Theorem 2.5, we get

$$\frac{1}{4}\left|a_{p+2}-\mu a_{p+1}^2\right| \le \begin{cases} (p+2)^2\left(\frac{(p+3)(2p+3)}{4(p+2)^2}-\mu\right), \text{if} \mu \le \frac{(p+3)(2p+3)}{4(p+2)^2}; \\ \frac{p+3}{4}, \text{if} \frac{(p+3)(p+1)}{2(p+2)^2} \le \mu \le \frac{p+3}{2(p+2)}\ ; \\ (p+2)^2\left(\mu-\frac{(p+3)(2p+3)}{4(p+2)^2}\right), \text{if} \mu \ge \frac{p+3}{2(p+2)} \end{cases}$$

which are the required results for the class $\left(S_p{}^*\right)^{-1}$(See [4].

Corollary 2.7: Putting $p = 1$, in Theorem 2.5, we get

$$\frac{1}{4\delta}|a_3-\mu a_2^2| \leq \begin{cases} (5\delta+1)-9\mu\delta, \text{if } \mu \leq \frac{(5\delta-1)}{9\delta}, \\ 2, \text{if} \frac{(5\delta-1)}{9\delta} \leq \mu \leq \frac{(5\delta-3)}{9\delta}\ ; \\ 9\mu\delta-(5\delta+1), \text{if } \mu \geq \frac{(5\delta-3)}{9\delta}. \end{cases}$$

which are the required results for the class $(S^*)^{-1}[\delta]$.(See [4])

Corollary 2.8: Putting $\delta = 1, p = 1$, in Theorem 2.5, we get

$$\frac{1}{4}|a_3-\mu a_2^2| \leq \begin{cases} (5-9\mu), \text{if } \mu \leq \frac{4}{9}; \\ 1, \text{if} \frac{4}{9} \leq \mu \leq \frac{2}{3}\ ; \\ (9\mu-5), \text{if } \mu \geq \frac{2}{3}. \end{cases}$$

which are the required results for the class $(S^*)^{-1}$.(See [4])

Theorem 2.9: If $\left(S_p{}^*\right)^{-1}[A,B;\delta]$, then

$$\frac{1}{\delta(A-B)}\left|a_{p+2}-\mu a_{p+1}^2\right|$$

$$\leq \begin{cases} \frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-B\right\}-\mu\delta(p+2)^2(A-B), \\ \text{if } \mu \leq \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-(1+B)\right\}}{\delta(p+2)^2(A-B)} & (2.27) \\ \frac{p+3}{2}, \\ \text{if} \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-(1+B)\right\}}{\delta(p+2)^2(A-B)} \leq \mu \leq \frac{\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-(1-B)\right\}}{\delta(p+2)^2(A-B)}\ ; & (2.28) \\ \mu\delta(p+2)^2(A-B)-\frac{p+3}{2}\left\{\left(p\delta+\frac{3\delta}{2}-\frac{1}{2}\right)(A-B)-B\right\}, \\ \text{if } \mu \geq \frac{2\left\{\frac{5\delta-1}{2}(A-B)-(1-B)\right\}}{9\delta(A-B)} & (2.29) \end{cases}$$

The results are sharp.

Proof: We have

$$\frac{zf(z)}{(p+1)\int_0^z f(z)dz} = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta}; \delta > 0 \qquad (2.30)$$

Expanding we have

$$\left(1+a_{p+1}z+a_{p+2}z^2+---\right)$$
$$= (1+\frac{p+1}{p+2}a_{p+1}z+\frac{p+1}{p+3}a_{p+2}z^2+---)(1+\delta(A-B)\,c_1z+\delta(A-B)\left[c_2+\left\{\frac{(\delta-1)}{2}(A-B)-B\right\}c_1{}^2\right]z^2--)$$

Identifying terms, we get

$$a_{p+1} = \delta(p+2)(A-B)c_1$$

and

$$a_{p+2} = \frac{p+3}{2}\,\delta(A-B)c_2 + \frac{p+3}{2}\,\delta(A-B)\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\}c_1^2 \qquad (2.31)$$

Using (2.31), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p+3}{2}\,\delta(A-B)c_2 + \frac{p+3}{2}\,\delta(A-B)\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\}c_1^2 - \mu(\delta(p+2)(A-B)c_1)^2$$

This gives

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p+3}{2}\,\delta(A-B)c_2 + \delta(A-B)\left[\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\} - \mu\delta(p+2)^2(A-B)\right]c_1^2$$

This gives

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{p+3}{2}|c_2| + \left|\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\} - \mu\delta(p+2)^2(A-B)\right||c_1^2|$$

$$\le \frac{p+3}{2} + \left[\left|\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\} - \mu\delta(p+2)^2(A-B)\right| - \frac{p+3}{2}\right]|c_1^2| \qquad (2.32)$$

Case I: $\mu \le \frac{\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\}}{\delta(p+2)^2(A-B)}$, we get from (2.32) that

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{p+3}{2} + \left[\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - (1+B)\right\} - \mu\delta(p+2)^2(A-B)\right]|c_1^2| \qquad (2.33)$$

Subcase I(a): $\mu \le \frac{\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - (1+B)\right\}}{\delta(p+2)^2(A-B)}$, Using $|c_1^2| \le 1$, (2.33) takes the form

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\} - \mu\delta(p+2)^2(A-B) \qquad (2.34)$$

Subcase I(b): $\mu \ge \frac{\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - (1+B)\right\}}{\delta(p+2)^2(A-B)}$. (2.33) takes the form

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{p+3}{2}. \qquad (2.35)$$

Case II: $\mu \ge \frac{\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\}}{\delta(p+2)^2(A-B)}$, we get from (2.32) that

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{p+3}{2} + \left[\delta(p+2)^2(A-B)\mu - \frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - (1-B)\right\}\right]|c_1^2| \qquad (2.36)$$

Subcase II (a): $\mu \le \frac{\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - (1-B)\right\}}{\delta(p+2)^2(A-B)}$. (2.36) takes the form

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{p+3}{2}. \qquad (2.37)$$

Subcase I(b): $\mu \ge \frac{\frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - (1-B)\right\}}{\delta(p+2)^2(A-B)}$. Using $|c_1^2| \le 1$, (2.36) takes the form

$$\frac{1}{\delta(A-B)}\left|a_{p+2} - \mu a_{p+1}^2\right| \le \mu\delta(p+2)^2(A-B) - \frac{p+3}{2}\left\{\left(p\delta + \frac{3\delta}{2} - \frac{1}{2}\right)(A-B) - B\right\} \qquad (2.38)$$

This completes the theorem. The results are sharp.

Corollary 2.10: Putting $\delta = 1$ in Theorem 2.9,we get

$$\frac{1}{A-B}\left|a_{p+2}-\mu a_{p+1}^{2}\right| \leq \begin{cases} (p+2)^2(A-B)\left(\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}-\mu\right), \\ \text{if } \mu \leq \frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)} ; \\ \frac{p+3}{2}, \\ \text{if } \frac{(p+3)[\{(p+1)(A-B)-B\}-1]}{2(p+2)^2(A-B)} \leq \mu \leq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)} \\ (p+2)^2(A-B)\left(\mu-\frac{(p+3)\{(p+1)(A-B)-B\}}{2(p+2)^2(A-B)}\right), \\ \text{if } \mu \geq \frac{p+3}{2}\frac{[\{(p+1)(A-B)-B\}+1]}{(p+2)^2(A-B)} \end{cases}$$

Which are the result of $f(z) \in \left(S_p{}^*\right)^{-1}[A,B]$, as proved in Theorem 2.1.

Corollary 2.11: Putting $A=1, B=-1$, in Theorem 2.9, we get

$$\frac{1}{2\delta}\left|a_{p+2}-\mu a_{p+1}^{2}\right| \leq \begin{cases} \delta\left\{\frac{p+3}{2}(2p+3)-2\mu(p+2)^2\right\}, \text{if } \mu \leq \frac{(p+1)(p+3)}{2(p+2)^2}, \\ \frac{p+3}{2}, \text{if } \frac{(p+1)(p+3)}{2(p+2)^2} \leq \mu \leq \frac{(p+3)(2p\delta+3\delta-3)}{4\delta(p+2)^2} ; \\ \delta\left\{2\mu(p+2)^2-\frac{p+3}{2}(2p+3)\right\}, \text{if } \mu \geq \frac{(p+3)\,(2p\delta+3\delta-3)}{4\delta(p+2)^2}. \end{cases}$$

which are the required results for the class $\left(S_p{}^*\right)^{-1}[\delta]$ as proved in Theorem 2.5.

Corollary 2.12: Putting $p=1$, in Theorem 2.9, we get

$$\frac{1}{\delta(A-B)}\left|a_3-\mu a_2^2\right| \leq \begin{cases} 2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-B\right\}-9\mu\delta(A-B), \\ \text{if } \mu \leq \frac{2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-(1+B)\right\}}{9\delta(A-B)}, \\ 2, \\ \text{if} \frac{2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-(1+B)\right\}}{9\delta(A-B)} \leq \mu \leq \frac{2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-(1-B)\right\}}{9\delta(A-B)} ; \\ 9\mu\delta(A-B)-2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-B\right\}, \\ \text{if } \mu \geq \frac{2\left\{\left(\frac{5\delta-1}{2}\right)(A-B)-(1-B)\right\}}{9\delta(A-B)}. \end{cases}$$

which are the required results for the class $(S^*)^{-1}[A,B;\delta]$ (See [4]).

Corollary 2.13: Putting $\delta=1, p=1$ in Theorem 2.9, we get

$$\frac{1}{A-B}|a_3-\mu a_2^2| \le \begin{cases} 9(A-B)\left(\frac{2(2A-3B)}{9(A-B)}-\mu\right), \text{if } \mu\le\frac{2(2A-3B)}{9(A-B)}; \\ \frac{p+3}{2}, \text{if } \frac{2(2A-3B)}{9(A-B)} \le \mu \le \frac{2(2A-3B+1)}{9(A-B)}\ ; \\ 9(A-B)\left(\mu-\frac{2(2A-3B)}{9(A-B)}\right), \text{if } \mu \ge \frac{2(2A-3B+1)}{9(A-B)} \end{cases}$$

Which are the result of $f(z) \in \left(S_p{}^*\right)^{-1}[A,B]$.

Corollary 2.14: Putting $\delta = 1, A = 1, B = -1,$ in Theorem 2.9, we get

$$\frac{1}{4}\left|a_{p+2}-\mu a_{p+1}^2\right| \le \begin{cases} (p+2)^2\left(\frac{(p+3)(2p+3)}{4(p+2)^2}-\mu\right), \text{if } \mu\le\frac{(p+3)(2p+3)}{2(p+2)^2(A-B)}; \\ \frac{p+3}{4}, \text{if } \frac{(p+3)(2p+3)}{2(p+2)^2(A-B)} \le \mu \le \frac{(p+3)(p+2)}{2(p+2)^2}\ ; \\ (p+2)^2\left(\mu-\frac{(p+3)(2p+3)}{4(p+2)^2}\right), \text{if } \mu \ge \frac{(p+3)(p+2)}{2(p+2)^2} \end{cases}$$

Which are the result of $\left(S_p{}^*\right)^{-1}$(See[4]).

Corollary 2.15: Putting $\delta = 1, A = 1, B = -1, p = 1$ in Theorem 2.9, we get

$$\frac{1}{4}|a_3-\mu a_2^2| \le \begin{cases} (5-9\mu), \text{if } \mu \le \frac{4}{9}; \\ 1, \text{if} \frac{4}{9} \le \mu \le \frac{2}{3}\ ; \\ (9\mu-5), \text{if } \mu \ge \frac{2}{3}. \end{cases}$$

which are the required results for the class $(S^*)^{-1}$.(See [4])

REFERENCES

[1] Chichra, P. N., "New subclasses of the class of close-to-convex functions", Procedure of American Mathematical Society, 62, (1977), 37-43.

[2] Goel, R. M. and Mehrok, B. S., "A subclass of univalent functions", Houston Journal of Mathematics, 8, (1982), 343-357.

[3] Goel, R. M. and Mehrok, B. S., "A coefficient inequality for certain classes of analytic functions", Tamkang Journal of Mathematics, 22, (1990), 153-163.

[4] Gurmeet Singh, Saroa M. S. and Mehrok, B. S., "Fekete-szegö inequality for a new class of analytic functions", Elsevier; Proc. Of International conference on Information and Mathematical Sciences, (2013), 90-93.

[5] Kaplan, W., "Close-to-convex schlicht functions", Michigan Mathematical Journal, 1, (1952), 169-185.

[6] Keogh, S. R. and Merkes, E. R., "A Coefficient inequality for certain classes of analytic functions", Proc. of American Mathematical Society, 20, (1989), 8-12.

[7] K. Löwner, "Uber monotone Matrixfunktionen", Math. Z., 38, (1934), 177-216.

[8] Kunle Oladeji Babalola, "The fifth and sixth coefficients of α-close-to-convex functions", Kragujevac J. Math., 32, (2009), 5-12.

[9] L. Bieberbach, "Uber einige Extremal probleme im Gebiete der konformen Abbildung", Math. Ann., 77(1916), 153-172.

[10] L. Bieberbach, "Uber die Koeffizientem derjenigen Potenzreihen, welche eine schlithe Abbildung des Einheitskreises vermitteln", Preuss. Akad. Wiss. Sitzungsb., (1916), 940-955.

[11] M. Fekete and G. Szegö, "Eine Bemerkung über ungerade schlichte Funktionen", J. London Math. Soc. 8 (1933), 85-89.

[12] R. M. Goel and B. S.Mehrok, "A coefficient inequality for a subclass of closeto-convex functions", Serdica Bul. Math. Pubs. 15, (1989), 327-335.

[13] Z. Nehari, "Conformal Mapping", McGraw-Hill, Comp., Inc., New York, (1952).